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TWO NUMERICAL METHODS FOR SOLVING NONLINEAR INTEGRAL EQUATION IN TWO-DIMENSIONAL PROBLEMS

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ABSTRACT

In this paper, the existence and uniqueness solution of nonlinear integral equations in two-dimensional problems is considered in the space $L_2(D)\times C(0,T)$*, where* D *is the domain of integration with respect to position, while* $t\in[0,T]$, $T<1$ *is the time. The equation takes a form of Fredholm- Volterra integral equation in nonlinear type (NF-VIE). Here, we represent the unknown function in the form of Chebeyshev and Legendre polynomials and then, using Collocation and Galerkin methods, as two numerical methods, the numerical solutions of the NF-VIE are obtained. Numerical results are computed and the error, in each case is calculated.*

KEYWORDS: *Two-Dimensional Problems- Nonlinear Fredholm –Volterra Integral Equation- Collocation And Galerkin Methods- Chebeyshev And Legendre Polynomials- Continuous Kernel*

INTRODUCTION

The first kind mixed integral equations can be solved analytically using one of these methods: Cauchy method, orthogonal polynomial method, Potential theory method and Krein's method. The importance of the first kind mixed integral equation and contact problem came from the work of Abdou in [1, 2]. The solution of its in one, two and three dimensional has been obtained, analytically using separation of variables method. Also, in [3, 4], the solution of integral equation of mixed type has been obtained numerically using Toeplitz matrix method and Nystrom method. Many different numerical methods are used to obtain the solution of mixed integral equation of the second kind two-dimensional problems, see [5-11]. In all previous and the following discussion, the Fredholm–Volterra integral equation is discussed in $L_2(D)\times C(0,T)$, where D is the domain of integration with respect to position, while $t\in[0,T]$, $T<1.$ is the time.

In this paper, we discuss and solve the nonlinear integral equation of the second kind, with continuous kernels, using Collocation and Galerkin methods [12, 13]. The solution of the integral equation is represented in the Chebeyshev polynomials and Legendre polynomials form in the space $L_2[0,1]\times C\left(0,T\right), T<1$. The kernel of Fredholm integral term is continuous kernel in position, while the kernel of Volterra integral term is a positive continuous function in time. Some numerical examples are computed and the error, in each case, is calculated.

THE EXISTENCE AND UNIQUENESS OF THE SOLUTION

Consider the Nonlinear Integral Equation

$$\varphi(x,t)=\gamma(x;t,f(x,t),\varphi(x,t))+\lambda\int_0^1 k(x,y)\varphi(y,t)dy+\lambda\int_0^t F(t,\tau)\varphi(x,\tau)d\tau \qquad (1)$$

The integral equation (1) is called nonlinear **F-V** integral type. This formula is considered in the space $L_2[0,1]\times C[0,T],T<1$. Here, the continuous kernel $k(x,y)$ of Fredholm integral term is considered in position. While the positive continuous kernel $F(t,\tau)$ of Volterra integral term is considered in time for all $t,\tau\in[0,T],T<1$

Assume the integral operator

$$W\phi=K\phi+F\phi \qquad (2)$$

where

$$K\varphi=\lambda\int_0^1 k(x,y)\varphi(y,t)dy,\quad F\varphi=\lambda\int_0^t F(t,\tau)\varphi(x,\tau)d\tau \qquad (3)$$

Hence, the formula (1) takes the form

$$\varphi=\gamma(f,\varphi)+W\varphi \qquad (4)$$

In order to guarantee the existence of a unique solution of Eq. (4), we assume the following conditions:

- The kernel of the Fredholm term $k(x,y)\in C\left([0,1]\times[0,1]\right)$, and satisfies the discontinuity condition:

  $$\left[\int_0^1\int_0^1 |k(x,y)|^2\,dy\,dx\right]^{\frac{1}{2}}=c,\quad (c \text{ is a constant})$$

- The kernel of the Volterra term $F(t,\tau)\in C\left([0,T]\times[0,T]\right)$, $,0\le\tau\le t\le T<1$, satisfies: $\left|F(t,\tau)\right|\le M,\ \forall t,\tau\in[0,T],\ (M-Const.)$

- The given function $\gamma(f(x,t),\varphi(x,t))$ with its partial derivatives with respect to x and t are continuous in $L_2[0,1]\times C[0,T]$, where

  $$\left|\gamma(x,t,f,\varphi)\right|\le 1+\alpha\left|\varphi\right|$$

  $$\|\gamma(f,\varphi)\|=\max_{0\le t\le T}\int_0^t\left[\int_0^1|\gamma(f,\varphi)|^2\,dx\right]^{\frac{1}{2}}d\tau=H,\ H \text{ is a constant}$$

- The known function $\gamma(f(x,t),\varphi(x,t))$, in the space $L_2[-1,1]\times C[0,T]$, $T<1$, satisfies Lipschitz condition with respect to the unknown function.

$$|\gamma(f,\varphi_1)-\gamma(f,\varphi_2)|\le A|\varphi_1-\varphi_2|.(A<1)$$

To prove the existence and uniqueness of the solution, we use Banish fixed point theorem. For this, we follow

The Normality of the Integral Operator (2.2), can be Proved as the Following

$$\|W\phi\|\le\|K\phi\|+\|F\phi\| \quad (5)$$

$$\|K\varphi\|\le|\lambda|\max_{0\le t\le T}\int_0^t\left[\int_0^1\left(\int_0^1|k(x,y)|^2dy\int_0^1|\varphi(y,t)|^2dy\right)dx\right]^{\frac{1}{2}}d\tau\le|\lambda|c\|\varphi\| \quad (6)$$

Also, we obtain

$$\|F\varphi\|\le|\lambda|\left\|\int_0^t|F(t,\tau)|\,\|\varphi(x,\tau)\|d\tau\right\|\le|\lambda|M\|\varphi\|T \quad (7)$$

Hence, we get

$$\|W\varphi\|\le|\lambda|(c+MT)\|\varphi\|\le\alpha\|\varphi\|,\quad \beta=|\lambda|(c+ML) \quad (8)$$

To Prove that the Solution is Continuous and Exist, We Assume the Two Functions

$\phi(x,t)$ and $\tilde{\phi}(x,t)$. Then, we write

$$\varphi(x,t)-\tilde{\varphi}(x,t)=[\gamma(f(x,t),\varphi(x,t))-\gamma(f(x,t),\tilde{\varphi}(x,t))]$$

$$+\lambda\int_0^1 k(x,y)[\varphi(y,t)-\tilde{\varphi}(y,t)]dy+\lambda\int_0^t F(t,\tau)[\varphi(x,\tau)-\tilde{\varphi}(x,\tau)]d\tau$$

This leads us to the following

$$\|\varphi(x,t)-\tilde{\varphi}(x,t)\|\le|\lambda|\left\|\int_0^1|k(x,y)|\,|\varphi(y,t)-\tilde{\varphi}(y,t)|\,dy\right\|$$

$$+|\lambda|\left\|\int_0^t|F(t,\tau)|\,|\varphi(x,\tau)-\tilde{\varphi}(x,\tau)|\,d\tau\right\|$$

Using condition (ii) then applying Cauchy – Schwarz inequality, we get

$$\|\varphi(x,t)-\tilde{\varphi}(x,t)\| \le |\lambda| \max_{0\le t\le T} \left| \int_0^t \left\{ \int_0^1\int_0^1 (k(x,y))^2 dx\,dy \right\}^{\frac{1}{2}} . \left\{ \int_0^1 [\varphi(y,t)-\tilde{\varphi}(y,t)]^2 dy \right\}^{\frac{1}{2}} d\tau \right|$$

$$+|\lambda| M \int_0^t \|\phi(x,\tau)-\tilde{\phi}(x,\tau)\| d\tau$$

Using condition (i) in the inequality, we obtain

$$\|\varphi(x,t)-\tilde{\varphi}(x,t)\| \le |\lambda| c \|\varphi(y,t)-\tilde{\varphi}(y,t)\| + |\lambda| M\,T \|\varphi(x,\tau)-\tilde{\varphi}(x,\tau)\|$$
$$\le \alpha \|\varphi(y,t)-\tilde{\varphi}(y,t)\|$$

Finally, we obtain

$$(1-\alpha)\|\phi(x,t)-\tilde{\phi}(x,t)\| \le 0 \ .$$

Since $\|\phi(x,t)-\tilde{\phi}(x,t)\|$ is necessarily non – negative, and $\alpha<1$, we get

$$\|\phi-\tilde{\phi}\|=0 \qquad \Rightarrow \quad \phi=\tilde{\phi} \tag{9}$$

It follows that if (1) has a continuous solution, which must be unique.

By the condition $\alpha<1$, $\overline{W}$ is a contraction operator, then it has a unique solution.

Collocation Method with Chebyshev and Legendre Polynomials

In this section, we present the all functions in the form of Chebyshev Polynomials and Legendre to obtain numerical solution of the **NF-VIE** of the second kind with continuous kernels.

Assume the **NF-VIE** of Eq. (1), and let

$$\varphi(u,t)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{n,m}\varphi_m(u)\varphi_n(t);\ f_n(u)\cong\sum_{n=0}^{N}\sum_{m=0}^{M} f_{n,m}\varphi_m(u)\varphi_n(t);\ k(u,v)=\sum_{l=0}^{M}\varphi_l(u)\varphi_l(v) \tag{10}$$

Where, $\varphi_l(...)$ are the eigenfunctions of degree l . The eigenfunctions, in this work, will be represented in the form of Chebyshev polynomials of the first order $T_n(...)$ **and** Legendre polynomials $P_n(...)$.

Using (10) in (1), and applying the collocation method, we follow

$$\sum_{m,n=0}^{\infty} a_{n,m}\varphi_m(u)\varphi_n(t) = \gamma_n(\sum_{m,n} f_{n,m}\varphi_m(u)\varphi_n(t), \sum_{m,n}^{\infty} a_{n,m}\varphi_m(u)\varphi_n(t))$$
$$+\lambda\int_0^t \sum_{\varsigma}^{Z}\sum_{m,n}^{\infty} a_{n,m}\varphi_\varsigma(\tau)\varphi_\varsigma(t)\varphi_n(t)\,\varphi_m(u)d\tau + \lambda\int_0^1 \sum_{1}^{L}\sum_{m,n}^{\infty} a_{n,m}\varphi_1(u)\varphi_n(t)\varphi_1(v)\,\varphi_m(v)dv. \quad (11)$$

Then

$$R_{M,N}(u,t) = \sum_{m,n=0}^{N,M} a_{n,m}\varphi_m(u)\varphi_n(t) - \gamma_n(\sum_{m,n} f_{n,m}\varphi_m(u)\varphi_n(t), \sum_{m,n}^{N,M} a_{n,m}\varphi_m(u)\varphi_n(t))$$
$$-\lambda\int_0^t \sum_{\varsigma}^{Z}\sum_{m,n}^{N,M} a_{n,m}\varphi_\varsigma(\tau)\varphi_\varsigma(t)\varphi_n(t)\,\varphi_m(u)d\tau - \lambda\int_0^1 \sum_{1}^{L}\sum_{m,n}^{N,M} a_{n,m}\varphi_1(u)\varphi_n(t)\varphi_1(v)\,\varphi_m(v)dv. \quad (12)$$

Where, $R_{M,N}(u,t)$ are the error of order (MxN), and vanishes at m points of position and n points of time, i.e . $R_{m,n}(u,t) = 0$

$$at\ \ 0 \le u_0 \le u_1 \le u_2 \le \ldots \le u_m = 1;\ \ 0 \le t_0 \le t_1 \le t_2 \le \ldots \le t_n = T$$

Galerkin Method with Chebyshev and Legendre Polynomials

The principal rule of Galerkin method is the error is orthogonal with the error in the space of integration. Assume $\varphi(u,t) = \sum_{n=0}^{\infty}\ \sum_{m=0}^{\infty} a_{n,m}\varphi_m(u)\varphi_n(t)$

Hence, we have

$$\sum_{m,n}^{M,N} a_{n,m}\int_0^1\int_0^t R_{m,n}(u,\tau).\varphi_m(u)\varphi_n(\tau)du d\tau = 0$$

i.e

$$\sum_{m,n=0}^{N,M} a_{n,m}\int_0^1\int_0^t \{\varphi_m(u)\varphi_n(t) - \gamma_n(\sum_{m,n} f_{n,m}\varphi_m(u)\varphi_n(t), \sum_{m,n}^{N,M} a_{n,m}\varphi_m(u)\varphi_n(t))$$
$$-\lambda\int_0^t \sum_{\varsigma}^{Z} \varphi_\varsigma(\tau)\varphi_\varsigma(t)\varphi_n(t)\,\varphi_m(u)d\tau \quad (13)$$
$$-\lambda\int_0^1 \sum_{1}^{L} \varphi_1(u)\varphi_n(t)\varphi_1(v)\,\varphi_m(v)dv\,\}\varphi_m(u)\varphi_n(\tau)du d\tau = 0$$

EXAMPLES AND NUMERICAL RESULTS

Example (1): Consider the following nonlinear integral equation

Table 1: The Error using Collocation Method When the Known Functions Take the Form of Chepyshev Polynomials and Legendre Polynomials at the Fixed Time t = 0.8

x	Exact Sol	Chepyshev Poly	Error of Cheby.	Legendre Poly	Error of Leg.
0.05	0.00160000000	0.00154035784	0.00005964215	0.00160065370	6.537035×10^{-7}
0.15	0.01440000000	0.01443041740	0.00003041740	0.01439315325	0.000006846741
0.2	0.02560000000	0.02567032184	0.00007032184	0.02558971219	0.000010287802
0.3	0.05760000000	0.05773988004	0.00013988004	0.05758344839	0.000016551602
0.4	0.10240000000	0.10259577066	0.00019577066	0.10237800902	0.000021990972
0.5	0.16000000000	0.16023799371	0.00023799371	0.15997339408	0.000026605911
0.6	0.23040000000	0.23066654917	0.00026654917	0.23036960358	0.000030396419
0.7	0.31360000000	0.31388143707	0.00028143707	0.31356663750	0.000033362498
0.8	0.40960000000	0.40988265738	0.00028265738	0.40956449585	0.000035504146
0.9	0.51840000000	0.51867021013	0.00027021013	0.51836317863	0.000036821364
0.1	0.64000000000	0.64024409529	0.00024409529	0.63996268584	0.000037314151

$$\varphi(x,t) = f(x,t) + 0.5\varphi^2(x,t) + \int_0^1 \sin(x+y)\varphi(y,t)dy + t\int_0^t \tau.\varphi(x,\tau)d\tau \quad (14)$$

$$(\varphi(x,t) = x^2 t^2; t = 0.8)$$

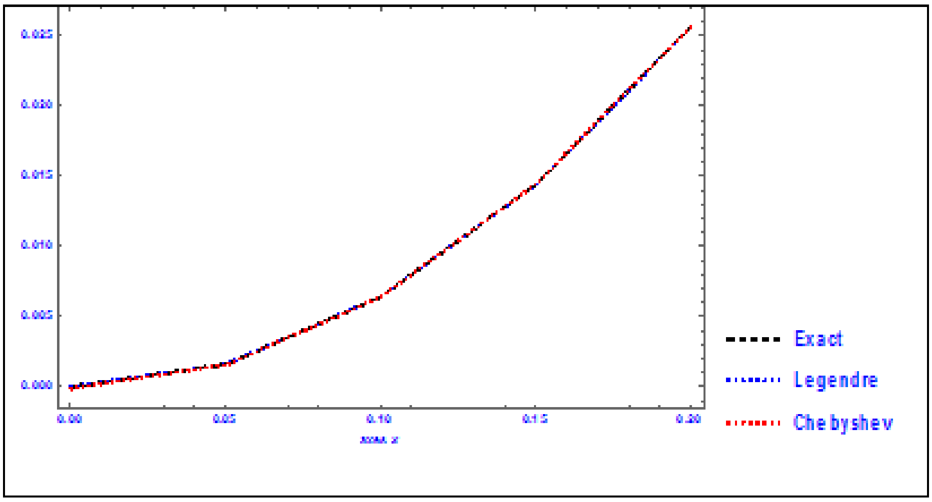

Figure 1: At t=0.8, μ=0.5.

Table 2: The Numerical Solution, Using Galerkin Method at the Fixed Time of t = 0.1

x	Exact Sol	Chebyshev Poly	Error of Chebyshev	Legendre Poly	Error of Legendre
0.05	0.0	0.00047235620	0.00047235620	0.000073942668	0.00007394266
0.15	0.0000250000	0.00024556506	0.00022056506	0.000020545461	0.000045545461
0.2	0.0001000000	0.00008537334	0.00001462665	0.000081394941	0.000018605058
0.3	0.0002250000	0.00000821893	0.00023321893	0.000231878538	0.000006878538
0.4	0.0004000000	0.00003521178	0.00043521178	0.000430905330	0.000030905330
0.5	0.0006250000	0.00000439479	0.00062060520	0.000678475317	0.000053475317
0.6	0.0009000000	0.00011060081	0.00078939918	0.000974588498	0.000074588498
0.7	0.0012250000	0.00028340626	0.00094159373	0.001319244875	0.000094244875
0.8	0.0016000000	0.00052281114	0.00107718885	0.001712444446	0.000112444446
0.9	0.0020250000	0.00082881546	0.00119618453	0.002154187212	0.000129187212
0.1	0.0025000000	0.00120141921	0.00129858078	0.002644473173	0.000144473173

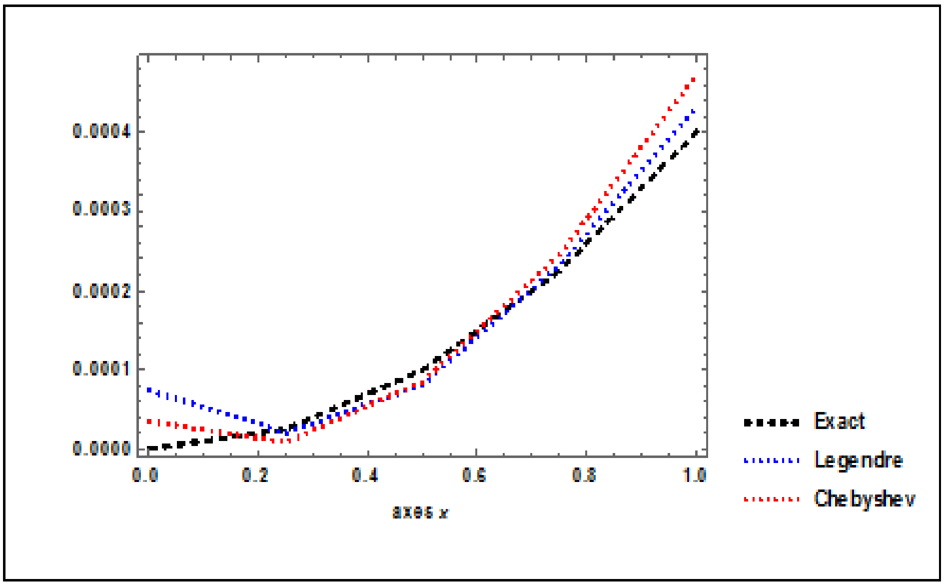

Figure 2: At t=0.1, μ=0.5.

Table 3: The Numerical Solution, Using Galerkin Method at the Fixed Time of t = 0.8

x	Exact Solution	Chebyshev Poly	Error of Chebyshev	Legendre Polynomial	Error of Legendre
0.05	0.0016000000	0.0016000000	1.1553×10^{-15}	0.0016000000	5.7246×10^{-16}
0.15	0.0144000000	0.0144000000	1.0755×10^{-15}	0.0144000000	5.4818×10^{-16}
0.25	0.0400000000	0.0400000000	1.0269×10^{-15}	0.0400000000	5.2736×10^{-16}
0.35	0.0784000000	0.0784000000	9.9920×10^{-16}	0.0784000000	4.4409×10^{-16}
0.45	0.1296000000	0.1296000000	8.6042×10^{-16}	0.1296000000	4.1633×10^{-16}
0.55	0.1936000000	0.1936000000	6.6613×10^{-16}	0.1936000000	2.7756×10^{-16}
0.65	0.2704000000	0.2704000000	4.9960×10^{-16}	0.2704000000	1.6653×10^{-16}
0.75	0.3600000000	0.3600000000	2.7755×10^{-16}	0.3600000000	1.2834×10^{-19}
0.85	0.4624000000	0.4624000000	1.1102×10^{-16}	0.4623999999	1.6653×10^{-16}
0.1	0.6400000000	0.6399999999	2.2204×10^{-16}	0.639999999	3.3307×10^{-16}

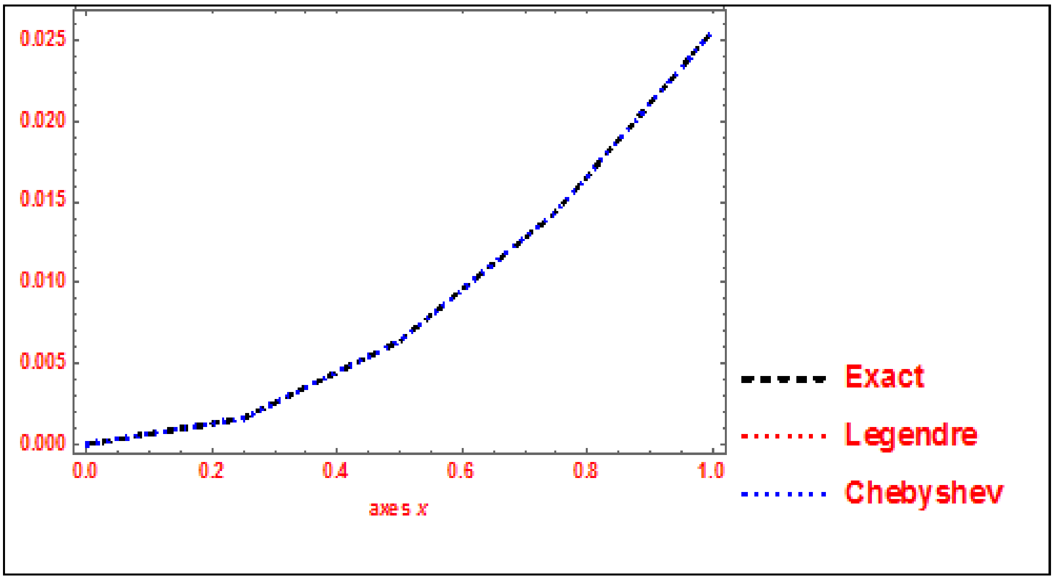

Figure 3: At t=0.8, μ=0.5.

Table 4: The Numerical Solution, Using Galerkin Method at the Fixed Time of t = 0.1

x	Exact Solution	Chebyshev Poly	Error of Chebyshev	Legendre Polynomial	Error of Legendre
0	0.0	2.8588×10^{-15}	2.8588×10^{-15}	2.0955×10^{-15}	2.0955×10^{-15}
0.1	0.0001000000	0.00010000000	2.8139×10^{-15}	0.0001000000	2.0610×10^{-15}
0.2	0.0004000000	0.00040000000	2.6914×10^{-15}	0.0004000000	1.9628×10^{-15}
0.3	0.0009000000	0.00090000000	2.5201×10^{-15}	0.0009000000	1.8288×10^{-15}
0.4	0.0016000000	0.00160000000	2.2698×10^{-15}	0.0016000000	1.6583×10^{-15}
0.5	0.0025000000	0.00250000000	1.9706×10^{-15}	0.0025000000	1.4320×10^{-15}
0.6	0.0036000000	0.00360000000	1.5933×10^{-15}	0.0036000000	1.1622×10^{-15}
0.7	0.0049000000	0.00490000000	1.1605×10^{-15}	0.0049000000	8.4220×10^{-16}
0.8	0.0064000000	0.00640000000	6.6092×10^{-16}	0.0064000000	4.7878×10^{-16}
0.9	0.0081000000	0.00810000000	8.1532×10^{-17}	0.0081000000	7.4644×10^{-17}
0.1	0.0100000000	0.00999999999	5.5164×10^{-16}	0.0099999999	3.9380×10^{-16}

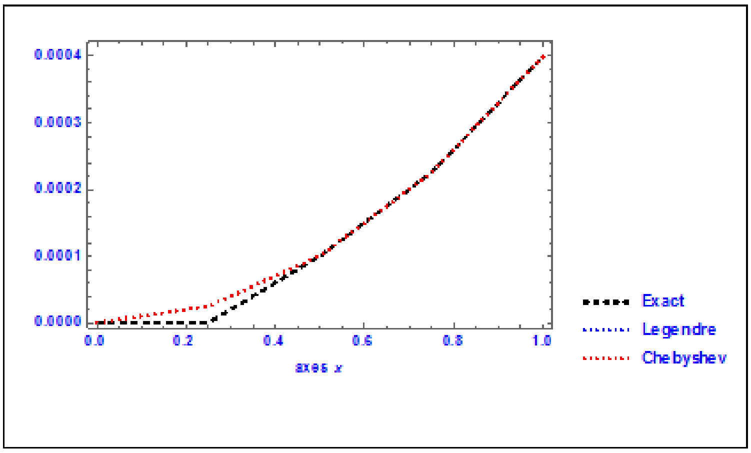

Figure 4: At t=0.1, μ=0.5.

Example (2): Consider the following nonlinear integral equation

$$\mu\varphi(x,t) = f(x,t) + \beta\varphi^2(x,t) - \lambda\int_0^1 ye^{2x}\varphi(y,t)dy - \lambda t\int_0^t t\tau.\varphi(x,\tau)d\tau \tag{15}$$

$$(\varphi(x,t) = x^2t^2;\ t = 0.8)$$

Table 5: Compared the Behaviour of the Error, Using Collection Method, for Chebyshev Polynomial and Legendre Polynomial at Time t = 0.8

x	Exact Sol	Chebyshev Poly	Error of Chebyshev	Legendre Poly	Error of Legendre
0	0	0.00014624302	0.000146243029	0.000009104795	0.000009104795
0.1	0.006400000000	0.006231670768	0.000168329231	0.006410058791	0.000010058791
0.2	0.025600000000	0.025407491813	0.000192508186	0.025610896051	0.000010896051
0.3	0.057600000000	0.057381220103	0.000218779896	0.057611616572	0.000011616572
0.4	0.102400000000	0.102152855640	0.000247144359	0.102412220357	0.000012220357
0.5	0.160000000000	0.159722398423	0.000277601576	0.160012707405	0.000012707405
0.6	0.230400000000	0.230089848452	0.000310151547	0.230413077715	0.000013077715
0.7	0.313600000000	0.313255205728	0.000344794271	0.313613331288	0.000013331288
0.8	0.409600000000	0.409218470249	0.000381529750	0.409613468123	0.000013468123
0.9	0.518400000000	0.517979642017	0.000420357982	0.518413488222	0.000013488222
1	0.640000000000	0.639538721032	0.000461278967	0.640013391583	0.000013391583

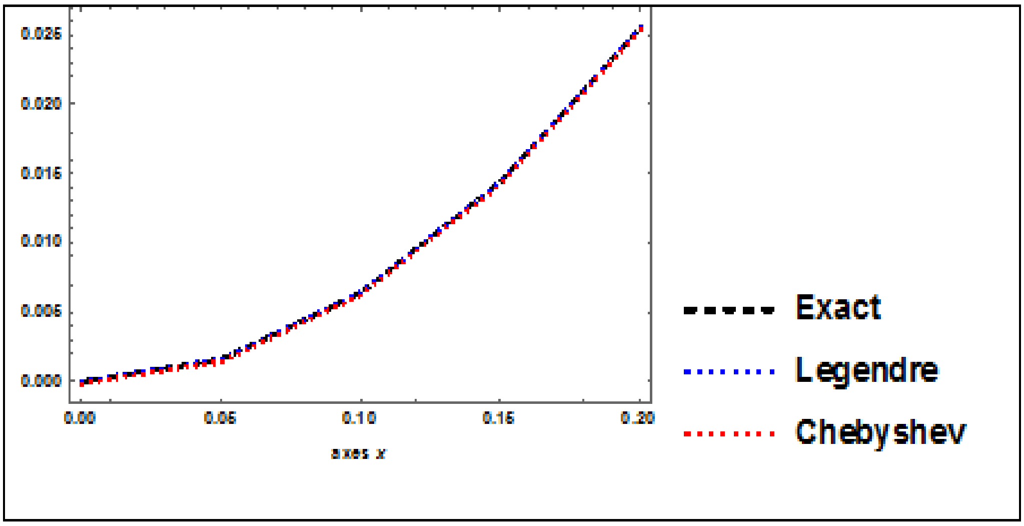

Figure 5: At t=0.8, μ=0.5.

Table 6: The Error for Chebyshev Polynomial and Legendre Polynomial, Using Collection Method at t = 0.1

x	Exact Solution	Chebyshev Polynomial	Error of Chebyshev	Legendre Polynomial	Error of Legendre
0. 5	0.000025000000	0.000646830497	0.000621830497	0.000062791076	0.000087791076
0.15	0.000225000000	0.001075883589	0.000850883589	0.000141363400	0.000083636599
0.25	0.000625000000	0.001703994682	0.001078994682	0.000544729500	0.000080270499
0.35	0.001225000000	0.002531163775	0.001306163775	0.001147307222	0.000077692777
0.45	0.002025000000	0.003557390868	0.001532390868	0.001949096566	0.000075903433
0.55	0.003025000000	0.004782675961	0.001757675961	0.002950097532	0.000074902467
0.65	0.004225000000	0.006207019055	0.001982019055	0.004150310121	0.000074689878
0.75	0.005625000000	0.007830420150	0.002205420150	0.005549734331	0.000075265668
0.85	0.007225000000	0.009652879244	0.002427879244	0.007148370164	0.000076629835
1	0.010000000000	0.012759801637	0.002759801637	0.009919845704	0.000080154295

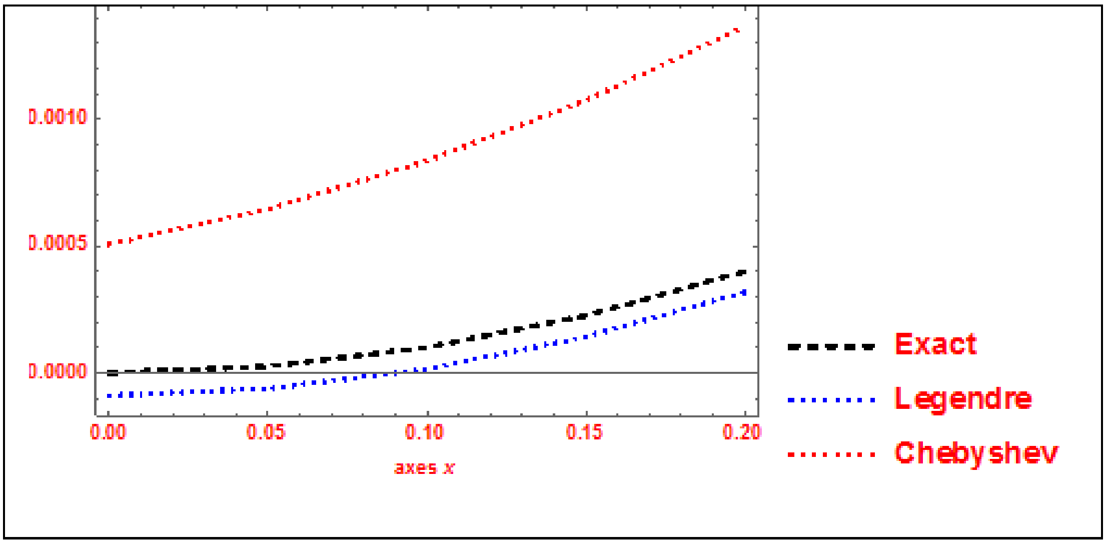

Figure 6: At t=0.1, μ=0.5.

Table 7: We Consider the Deference between the Error of Legendre Polynomial and Chebyshev Polynomial using Galerkin Method at Time t = 0.8

x	Exact Solution	Chebyshev Polynomial	Error of Chebyshev	Legendre Polynomial	Error of Legendre
0.05	0.00160000000	0.0016000000000	8.222589×10^{-16}	0.001600000000	3.78169×10^{-16}
0.15	0.01440000000	0.0144000000000	7.979727×10^{-16}	0.014400000000	4.09394×10^{-16}
0.25	0.04000000000	0.0400000000000	7.494005×10^{-16}	0.040000000000	3.60822×10^{-16}
0.35	0.07840000000	0.0784000000000	7.216449×10^{-16}	0.078400000000	3.05311×10^{-16}
0.45	0.12960000000	0.1296000000000	6.106226×10^{-16}	0.129600000000	2.49800×10^{-16}
0.55	0.19360000000	0.1936000000000	4.718447×10^{-16}	0.193600000000	1.94289×10^{-16}
0.65	0.27040000000	0.2704000000000	3.330669×10^{-16}	0.270400000000	1.11022×10^{-16}
0.75	0.36000000000	0.3600000000000	2.220446×10^{-16}	0.360000000000	0.
0.85	0.462400000000	0.4624000000000	1.110223×10^{-16}	0.462399999999	1.11022×10^{-16}
1	0.640000000000	0.6399999999999	2.220446×10^{-16}	0.639999999999	3.33066×10^{-16}

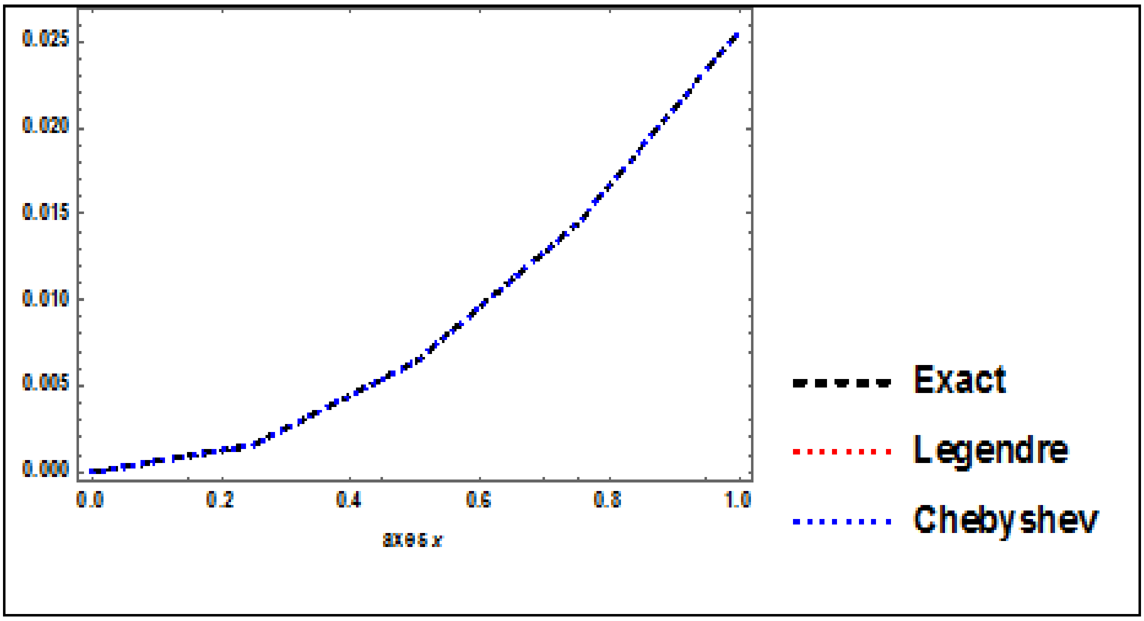

Figure 7: At t=0.8, μ=0.5.

Table 8: The Error of Galerkin Method using Legendre Polynomial and Chebyshev Polynomial Time t = 0.1					
x	**Exact Solution**	**Chebyshev Poly**	**Error of Chebyshev**	**Legendre Polynomial**	**Error of Legendre**
0	0	5.5511×10^{-16}	5.5511×10^{-16}	5.6898×10^{-16}	5.6898×10^{-16}
0.1	0.0001000000	0.0001000000	5.6660×10^{-16}	0.00010000000	5.5402×10^{-16}
0.2	0.0004000000	0.0004000000	5.0046×10^{-16}	0.00040000000	5.3039×10^{-16}
0.3	0.0009000000	0.0009000000	4.6794×10^{-16}	0.00090000000	5.1304×10^{-16}
0.4	0.0016000000	0.0016000000	4.3975×10^{-16}	0.00160000000	4.7271×10^{-16}
0.5	0.0025000000	0.0025000000	3.7730×10^{-16}	0.00250000000	4.3237×10^{-16}
0.6	0.0036000000	0.0036000000	3.2005×10^{-16}	0.00360000000	3.8120×10^{-16}
0.7	0.0049000000	0.0049000000	2.5240×10^{-16}	0.00490000000	3.1832×10^{-16}
0.8	0.0064000000	0.0064000000	1.7954×10^{-16}	0.00640000000	2.5847×10^{-16}
0.9	0.0081000000	0.0081000000	8.5001×10^{-17}	0.00810000000	1.8041×10^{-16}
0.1	0.0100000000	0.0099999999	1.9081×10^{-17}	0.01000000000	1.0408×10^{-16}

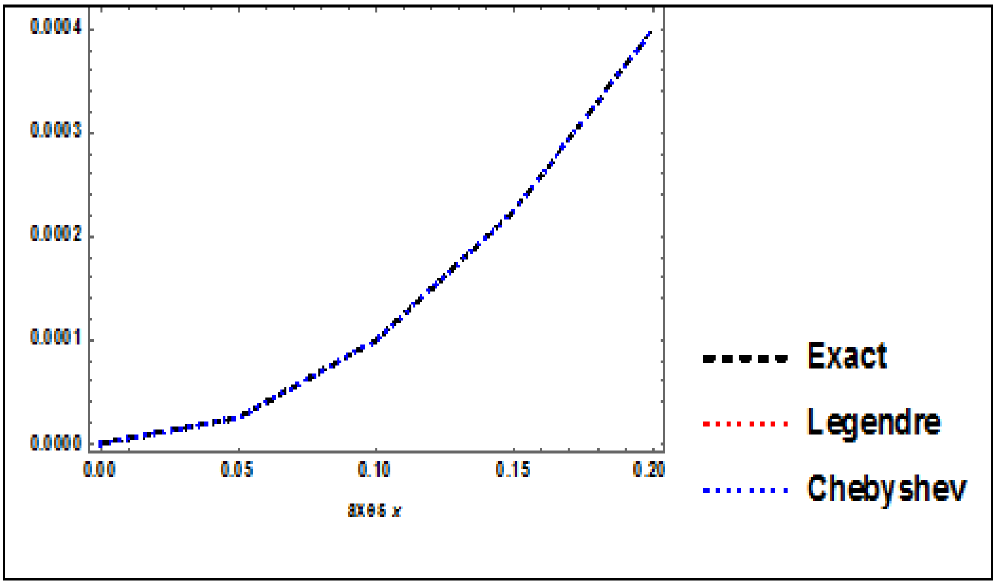

Figure 8: At t=0.1, μ=0.5.

CONCLUSIONS

Here, we consider a nonlinear integral equation of the second kind with continuous kernels.

Then, we obtain the numerical solution of the **NF-VIE** using collection method and Galerkin method. Whereas, the functions of the integral equation are represented in the form of Legendre polynomial and Chebyshev polynomial. The error, in each example is computed.

In Example 1, After Using Collection Method We Note That

In table (1), and at t = 0.8, with using Chebyshev polynomial on the interval $0 \le x \le 1$, the behaviour of the error is almost stable. While, at Legendre polynomial on the interval of $0 \le x \le .05$, the error increasing, and starting decreasing through the interval $0.1 \le x \le 1$.

In table (2), and t = 0.1, after using Chebyshev polynomial on the interval $0 \le x \le 0.3$, the error is decreasing, and increasing on the interval $0.31 \le x \le 1$. While, for the Legendre polynomial the error decreasing on the interval between $0.0 \le x \le 0.35$, then starting increasing in the interval, $0.36 \le x \le 1$

In Example 1, After Using Galerkin Method We Note That

In table (3), and at t = 0.8, the error behaviour after using Chebyshev polynomial in the interval $0 \le x \le 1$, is highly decreasing with stable result. Also, the error behaviour, in the same interval, of the Legendre polynomial is high stability of decreasing. The application of Chebyshev polynomial is better than the Legendre polynomial.

In table (4) at t = 0.1: the error after using Chebyshev polynomial and Legendre polynomial is stable. But from the results, we note that the error behaviour after using of Chebyshev polynomial is better than the error behaviour after using Legendre polynomial.

In Example 2, After Using Collection Method We Have

In table (5), and at t = 0.8, the error of Chebyshev polynomial in $0 \le x \le 1$ is stable. While, the error of Legendre polynomial on the interval of $0 \le x \le 0.05$ the error increasing, and start to decreasing on the interval $0.5 < x \le 1$

In table (6), and at t = 0.1, the error of Chebyshev polynomial in the interval $0 \le x \le 0.2$ is decreasing, and increasing in the interval $0.2 < x \le 1$. While at Legendre polynomial the error almost stable on the interval $0 \le x \le 1$.

Moreover, from the numerical results we note that the error behaviour of Chebyshev polynomial at t = 0.55 in the interval $0 \le x \le 0.5$ is increasing while, in $0.55 < x \le 1$ is decreasing. While, the error of the Legendre polynomial on the interval of $0 \le x \le 1$ is stable

In Example 2, After Using Galerkin Method We Have

In table (7), and in table (8) at t = 0.8 and t= 0.1 respectively, the error behaviour of Chebyshev polynomial and Legendre polynomial is stable in $0 \le x \le 1$.

We conclude that we have a good accuracy with Galekin method to solve Fredholm-Volterra integral equation.

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